数学花园

探秘

MATHEMATICS GARDEN

2023 Spring Cup

Mathematical Olympiad

PRELIMINARY ROUND

Date： 28 January 2023

Time Given： 1 hour 30 minutes

Level: Primary 6

Name: ____________

Instructions to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer ALL 20 questions.
3. Write your answers in the answer sheet provided.
4. No steps are needed to justify your answers.
5. Questions 1-7 are worth 4 marks each.
6. Questions 8-14 are worth 6 marks each.
7. Questions 15-19 are worth 8 marks each.
8. Questions 20 is worth 10 marks.
9. No marks will be deducted for wrong answers.
10. No marks will be given for unanswered questions.
11. No calculators or mathematical instruments are allowed.

Questions 1 to 7 are worth 4 marks each.

1. A new operation symbol ※ is defined as a※b= $3a-2b$. If x※(4※1)=7, then x=__________.

【Solution】We can first calculate $4※1=3\times4-2\times1=10$, it follows that x※(4※1)= x※10=3x-20. From there we get 3x-20=7 and so x=9.

2. The numbers 1 to 65 are arranged according to the figure below. After removing the multiples of 6, there are 55 numbers left. What is the sum of the remaining 55 numbers?

1	7	13	19	25	31	37	43	49	55	61
2	8	14	20	26	32	38	44	50	56	62
3	9	15	21	27	33	39	45	51	57	63
4	10	16	22	28	34	40	46	52	58	64
5	11	17	23	29	35	41	47	53	59	65

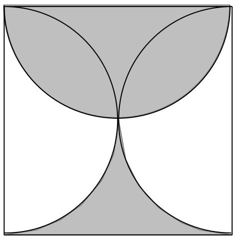
【Solution】Subtract the sum of 1 to 65 by the sum of the sequence 6，12，18，···，60:

$$(1+65)\times65\div2-(6+60)\times10\div2=2145-330=1815$$

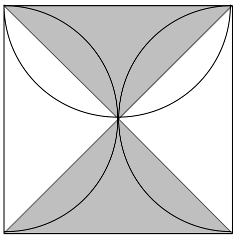
3. In a charity, Adam, Bob and Carla donated 80 dollars in total. If Adam donated 18 dollars more than Carla and the ratio between the sum of donation of Adam and Bob, and the sum of donation of Bob and Carla is 10 : 7, how much did Carla donate?

【Solution】Since Adam donated 18 dollars more than Carla, it follows that the sum of Adam and Bob's donation is 18 dollars more than the sum of Bob and Carla's donation. So, the sum of Adam and Bob's donation is $18\div(10-7)\times10=60$ dollars while the sum of Bob and Carla's donation is $60-18=42$ dollars. Therefore, Adam has donated $80-42=38$ dollars, Bob has donated $60-38=22$ dollars and Carla has donated $38-18=20$ dollars.

4. As shown in the figure, three semicircles were drawn in a square with a side of 2cm. If the diameter of the semicircles is the side of the square, the area of the shaded region is ______ cm^2.



【Solution】We can apply the shifting technique here by cutting out two arcs from the shaded semicircle and shift it downwards to construct an isosceles triangle.



The area of the two shaded triangles are identical and their total area is simply half of the area of the square which is $2^2 \times \frac{1}{2} = 2$ centimeter squared.

5. How many ways are there to pick two numbers from numbers 1 to 40 such that the sum of the two numbers is divisible by 4?

【Solution】The numbers 1 to 40 can be categorized into three types according to their remainder when divided by 4:

First type: Remainder is 0. There are ten numbers from 1 to 40 is divisible by 4 and there are $10 \times 9 \div 2 = 45$ ways to pick two numbers from these ten numbers.

Second type: Remainder is 1 or 3. There are ten numbers from 1 to 40 with remainder 1 and another ten numbers with remainder 3, so a total of $10 \times 10 = 100$ ways to pick two numbers from them.

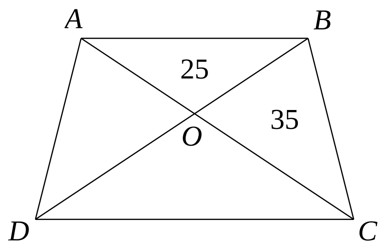
Third type: Remainder is 2. Similar to the first type, there are 45 ways to pick two numbers.

Therefore, there are $45 + 100 + 45 = 190$ ways in total.

6. Alice bought 10 kg of mushrooms. The mushrooms contain 99% water. After putting it under the sun, the mushrooms now contain 98% water. The weight of water that has evaporated is _______ kg.

【Solution】Only the water is evaporated but the other substances in the mushroom remains unchanged. There are $10\times(1-99\%)=0.1$kg of substances (excluding water) in the mushrooms originally, after placing them under the sun, there are still 0.1kg of substances (excluding water) in the mushroom, so $0.1\div(1-98\%)=5$kg of water evaporated.

7. As shown in the figure, $ABCD$ is a trapezium with two parallel lines AB and CD, the diagonal AC and BD intersects at point O. If the area of $\triangle AOB$ and $\triangle BOC$ is 25 cm^2 and 35 cm^2 respectively, the area of trapezium $ABCD$ is __________ cm^2



【Solution】According to the butterfly model, we have $S_{\triangle AOB}:S_{\triangle BOC}=a^2:ab=25:35$, it follows that $a:b=5:7$. Applying the butterfly model again, we get $S_{\triangle AOB}:S_{\triangle DOC}=a^2:b^2=5^2:7^2=25:49$ so $S_{\triangle DOC}=49$ (centimeter squared). Therefore, the area of $ABCD$ is $25+35+35+49=144$ (centimeter squared).

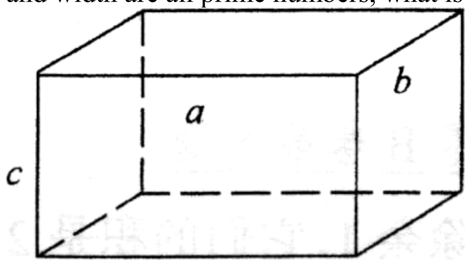
<u>Questions 8 to 14 are worth 6 marks each.</u>

8. Jockey is writing some digits after the number 1989, he follows the rule that the next number he writes down is the ones place of the product of the previous two numbers (for example, $8\times9=72$, so Jockey writes down 2 after 9, $9\times2=18$, so Jockey writes down 8 after 2). Since the first 4 digits are 1, 9, 8 and 9, from left to right, what is the sum of the first 1999 digits?

【Solution】

According to the question, we write out the first few digits: 19892868842868842…… and noticed that "286884" repeats with period six. To find the 1999th digit, since $(1999-4)\div 6=332\cdots 3$, the 1999th digit is 6. Therefore, the sum of first 1999 digits is $(1+9+8+9)+(2+8+6+8+8+4)\times 332+(2+8+6)$ $=27+11952+16$ $=11995$.

9. As shown in the figure, the sum of the area of a cuboid's front and top is 209. If its length, height and width are all prime numbers, what is the volume of this cuboid?



【解析】

The sum of the area of front and top is ac+ab=209 which can be expressed as ac+ab=a×(c+b)=209. Since 209=11×19, we can have a=11, this means c+b=19, when the sum of two prime numbers is odd, one of the prime number must be 2, so c+b=2+17. On the other hand, when a=19, c+b=11, so c+b=2+9, b is not a prime number. Therefore, the product is 11×2×17=374.

10. $\dfrac{\frac{1}{2}}{1+\frac{1}{2}}+\dfrac{\frac{1}{3}}{(1+\frac{1}{2})\times(1+\frac{1}{3})}+\cdots+\dfrac{\frac{1}{1999}}{(1+\frac{1}{2})\times(1+\frac{1}{3})\times\cdots\times(1+\frac{1}{1999})}=$__________

【Solution】 $\dfrac{\frac{1}{n+1}}{(1+\frac{1}{2})\times(1+\frac{1}{3})\times\cdots\times(1+\frac{1}{n+1})}=\dfrac{\frac{1}{n+1}}{\frac{n+2}{2}}=\dfrac{2}{(n+1)(n+2)}=2\times(\dfrac{1}{n+1}-\dfrac{1}{n+2})$

We can express the sum of the above sequence as: $\left[(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+\cdots+(\frac{1}{1999}-\frac{1}{2000})\right]\times 2=1-\dfrac{1}{1000}=\dfrac{999}{1000}$

11. Three remainders can be obtained after dividing a whole number N by 70, 110 and 160, if the sum of the three remainders is 50, what is the number N?

【Solution】Any number that satisfies the conditions in the question is acceptable, such as 660.

12. Student A, B, C, D and E entered a drawing competition. After the competition, student A and B went to ask the judge about the results, the judge told student A: “Unfortunately, you and student B are not the champion” and he told student B “You’re not in the last place”. Based on this information, how many possible ways are there to rank the five students?

【Solution】From the question, we know that if we line the five students up according to their ranking, A and B cannot be in the first place while B cannot be in the last place. Since there are more restrictions on placing student B, we need to consider him first, there are 3 ways to place B. After that, there are also 3 ways for student A and the other three students can be placed randomly with $P_3^3=3\times2\times1=6$ different ways. From the counting principle, there are $3\times3\times6=54$ ways of arrangements in total.

13. There are 25 tokens labelled with an odd number from 1 to 49. Two pupils each take turns to take a token. If one of them take the token with label x, the next person must take the token labelled with the greatest odd number factor of $99-x$. If the first pupil takes the token with label 5, how many tokens would be left when the game ends?

【Solution】If

x	$99-x$
5	47
47	13
13	43
43	7
7	23
23	19
19	5

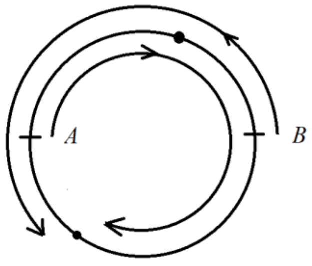
When the first pupil take the coin with label 19, the next pupil must take the coin with label 5 and the game ends. Since a total of 7 coins have been drawn, there will be $25-7=18$ coins left.

14. 100 cards were placed on the floor written with the number 1 to 100 and none of the number repeats. At least how many cards must be drawn to ensure the product of the number on the drawn cards is divisible by 12?

【Solution】From the number 1 to 100, there are $\left[\frac{100}{3}\right]=33$ multiple of three, which means there are $100-33=67$ numbers that are not multiple of 3. Since $12=2^2\times3$, if we take these 67 numbers, we cannot guarantee the product is a multiple of 3, however if we take 68 numbers, we must have at least one multiple of three among the 68 numbers. On top of that, since there are 50 odd numbers between 1 to 100, there are at least 18 even numbers in these 68 numbers, guarantee the product is also a multiple of 4. Therefore, with 68 numbers (68 cards) we can guarantee the product is a multiple of 12.

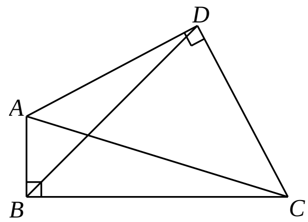
Questions 15 to 19 are worth 8 marks each.

15. Aaron and Ben started running from two opposite points on the diameter of a circular running track. Aaron runs in the clockwise direction and Ben runs in the anti-clockwise direction. When Ben has ran for 100 meters, they meet for the first time. They meet for the second time when Aaron left 60 meters to reaching his starting point. What is the perimeter of this circular track?

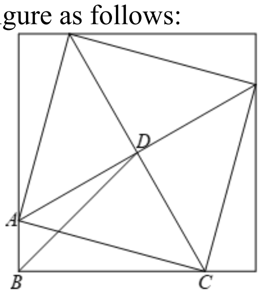


【Solution】From the figure, we see that when Aaron and Ben first met, Aaron and Ben have just finished running $\frac{1}{2}$ of the circular track. When they meet each other for the second time, they have finished running $1+\frac{1}{2}=\frac{3}{2}$ of the circular track. This tells us that the time it takes to meet each other for the first and second time is 1:3. It follows that the distance travelled by Ben on the second time he meet Aaron is 3 times the first, 100×3=300 meters. Finally, from the fact that Aaron and Ben have travelled (1 circular track - 60)+300 on the second time they meet which is also $\frac{3}{2}$ of the circular track, the perimeter of the circular track is 480 meters.

16. As shown in the figure below, $\angle ABC$ and $\angle ADC$ are two right angles in the quadrilateral *ABCD*. If $AD = DC$, $BD = 6$ and $AC = 8$, find the area of the triangle *ABC*.



【Solution】If we rotate the figure by 90, 180 and 270 degrees around point D, we get a new figure as follows:



Since $BD = 6$, the area of the outer square can be found to be $6\times 6\times 2 = 72$. Since $AC = 8$, the area of the smaller square is $8\times 8 = 64$. From there, we can find the area of the four small right angle triangles are $72 - 64 = 8$, so the area of triangle *ABC* is $8\div 4 = 2$.

17. Port A and B are 15 km apart along a river. Port A is upstream and port B is downstream. A ferry and a boat start travelling downstream from port A and port B respectively at the same time. After 5 hours, the ferry catches up with the boat. After travelling for another 1 hour, the ferry dropped an item into the river (the item floats on the river). The crew on the ferry only realized this after 6 minutes and they quickly made a turn to look for the item. When they reach the item, they meet again with the boat. What is the speed of the boat in km/h?

【Solution】

This question can be broken down into four stages:

First stage: Since it takes the ferry 5 hours to travel 15 km to catch up with the boat, we know their speed difference is 15÷5=3 km/h. Because they are both travelling downstream, their speed difference in still water is also 3 km/h.

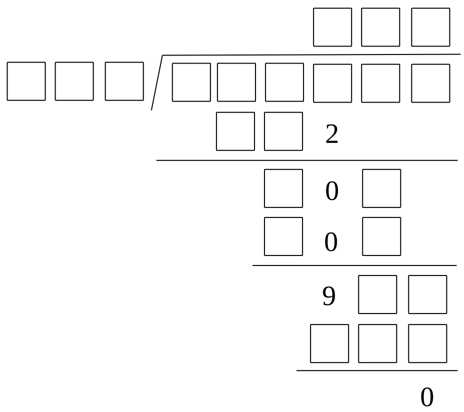
Second stage: In the following 1 hour, the ferry travels ahead of the boat until the point where they are separated by 3×1=3 km and an item was dropped by the ferry. The crew on the ferry only noticed this after 6 minutes and the distance between the ferry and the item now is "ferry's speed in still water ×1/10 km".

Third stage: The ferry takes a turn to retrieve the item is another catching-up problem. The sum of the speed of the ferry and the item equals the speed of the ferry in still water, so it takes "speed of the ferry in still water×1/10÷ speed of the ferry in still water = 1/10 hour" to retrieve the item.

Forth stage: When the ferry begins travelling back to retrieve the item, the ferry and the boat are separated by 3+3*1/10=33/10 km. They took 1/10 hour to meet each other at the location of the item, meaning that the sum of their speed is 33/10÷1/10=33 km/h which is also the sum of their speed in still water.

Finally, since we know the speed of the ferry is 3 km/h faster than the boat in still water, the speed of the boat is thus (33-3)÷2=15 km/h.

18. Complete the following vertical algorithm by filling in the blanks. If we know the quotient is an odd number, what is the dividend?



【Solution】

Focusing on the second and third row, we see that when a 4-digit number is subtracted by a 3-digit number, we get a 2-digit number, this means the first and second digit of the 4-digit number must be 1 and 0. From here, we also know that the first digit of the 3-digit number must be 9 and the last digit cannot be 0. On the other hand, since the divisor is a 3-digit number, when we multiply it with the hundreds place and the ones place of the quotient, the hundreds place of the two products are both 9, this tells us the hundreds place and the ones place of the quotient are the same. By filling in the known information so fat and label the unknown with alphabets, we get the following figure:

$$\begin{array}{ccc|cccccc}
 & & & & & & \boxed{e} & \boxed{f} & \boxed{e} \\ \hline
\boxed{a} & \boxed{b} & \boxed{c} & \boxed{1} & \boxed{0} & \boxed{\ } & \boxed{2} & \boxed{\ } & \boxed{2} \\
 & & & & \boxed{9} & \boxed{d} & 2 & & \\ \hline
 & & & & & \boxed{\ } & 0 & \boxed{\ } & \\
 & & & & & \boxed{\ } & 0 & \boxed{\ } & \\ \hline
 & & & & & & 9 & \boxed{d} & \boxed{2} \\
 & & & & & & \boxed{9} & \boxed{d} & \boxed{2} \\ \hline
 & & & & & & & & 0
\end{array}$$

Since the question tells us the quotient is an odd number, e must be an odd number (1, 3, 7, 9 but not 5).

If e is 1, $\overline{abc}=\overline{9d2}$ and $\overline{abc}\times f=\overline{9d2}\times f$ must be a 3-digit number, then $f=1$, but this leads to contradiction as the tens place of the product cannot be 0 and d cannot be 0.

If e is 3, $\overline{abc}=\overline{9d2}\div 3$ so d can be 1, 4, 7 with $\overline{abc}$ as 304, 314, 324 accordingly. When $\overline{abc}$ is 314 and 324, the tens place of product $\overline{abc}\times f$ is not 0 leads to inconsistent. When $\overline{abc}$ is 304, then f can be 1 or 2, this again leads to inconsistent so e cannot be 3.

If e is 7, $\overline{abc}=\overline{9d2}\div 7$ is satisfied when $d=5$. In this case, $\overline{abc}=136$ so $f=3$ which is consistent with the question, e can be 7.

If e is 9, $\overline{abc}=\overline{9d2}\div 9$, here d can only be 7 which gives us $\overline{abc}=108$. This leads to contradiction because f is 1.

So, we know the divisor must be 136 and from the quotient $\overline{efe}=737$, the dividend is $737\times 136=100232$

19. At least how many cubes with edge of length 1 does it take to form a cuboid with surface area 52?

【Solution】The more layers of cubes we have, the more squares coincide with each other, so the rule of thumb here is to have as little layers as possible.

(1) $1\times 1\times a$

With a height of 1, width of 1 and length of a, the surface area of the cuboid is $(1\times 1+1\times a+1\times a)\times 2=4\times a+2$. When $4\times a+2=52$, a is a decimal, so this type of cuboid is not possible.

(2) $1\times a\times b$

With a height of 1, width of a and length of b, the surface area of the cuboid is $(1\times a+1\times b+a\times b)\times 2=2ab+2a+2b=52$ which gives $a\times(b+1)+b=26$. When $b=2$, $a=8$, there are $a\times b=16$ cubes. From the property of prime numbers, when $b=3$、7、9、13、15、17、19、21、23、24, a does not exist. When $b=4$, a does not exist. When $b=5$, a does not exist. When $b=8$, $a=2$. Since a is greater than 1, $a\times b+a+b$ is greater than $3\times b$. Therefore, when b is greater than 9, $a\times b+a+b$ is greater than 27, since $a\times b+a+b=26$ is not possible, we need at least 16 cubes.

Questions 20 is worth 10 marks.

In your opinion, from question 1 to 19, your favourite question is question ________ and the most difficult question is question ________.

(As long as your answer is within 1 to 19, you get full marks, otherwise you get zero.)