RMO Round 1 2024 (Provided by Kangaroo Study students)

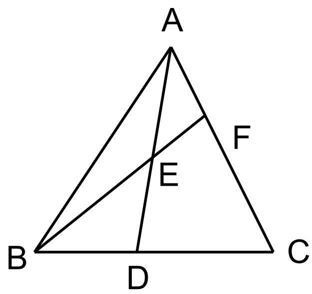
1. Given that a_n is the remainder of $7^n \div 11$, find $a_1 + a_2 + a_3 + ... + a_{2024}$.

[Answer] 11127

[Solution]

Remainders when 7, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9, 7^{10} is divided by 11 are 7, 5, 2, 3, 10, 4, 6, 9, 8, 1 respectively and this pattern repeats. Every 10 numbers, the sum of the remainders is $7+5+2+3+10+4+6+9+8+1=55$. Since $2024 \div 10 = 202R4$, we have 202 groups of 10 numbers and 4 remaining, it follows the answer is $202 \times 55 + 7 + 5 + 2 + 3 = 11127$.

2. Given that $BD : DC = 3 : 4$ and E is the mid-point of AD. Given that the sum of the area of ΔAEF and ΔEBD is 30, find the area of ΔABC.



[Answer] 100

[Solution]

Let $\Delta ABC = U$

$\Delta BED = U \times \frac{3}{7} \times \frac{1}{2} = \frac{3}{14}U$, $\Delta ABE = \Delta BED = \frac{3}{14}U$, $\Delta BEC = \frac{3}{14}U \div 3 \times 7 = \frac{1}{2}U$, $\Delta ABE : \Delta BEC = \frac{3}{14} : \frac{1}{2} = 3:7$,

$AF : FC = 3:7$, $\Delta AEF = U \times \frac{4}{7} \times \frac{1}{2} \times \frac{3}{10} = \frac{3}{35}U$, $\frac{3}{14}U + \frac{3}{35}U = \frac{15}{70}U + \frac{6}{70}U = \frac{3}{10}U$, $30 \div \frac{3}{10} = 100$

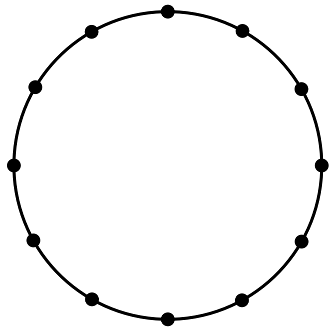
3. Student A and Student B set off from points X and Y, respectively, simultaneously, moving towards each other. If Student A increases his speed by 15% and Student B increases his speed by 12 km/h, they would meet at the same location as if they hadn't increased their speeds. Determine the speed of Student B before they increased their speed.

[Answer] 80 km/h

[Solution]

$12 \div 15\% = 12 \div \frac{3}{20} = 12 \times \frac{20}{3} = 80$ km/h

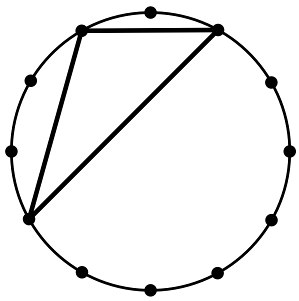
4. As shown in the figure below, there are 12 points on a circle. How many distinct triangles can be created using any 3 points as vertices, considering triangles that are identical upon rotation and flipping as one triangle?



[Answer] 12

[Solution]

The three sides of a triangle can be created by passing through any combination of the 12 points. Therefore, the various ways of partitioning the number 12 into the sum of 3 numbers correspond to the number of triangles that can be formed with different side lengths. For instance, [2, 3, 7] corresponds to the triangle below:



There are a total of 12 ways to split 12 into sum of 3 numbers, namely:

$12=1+1+10=1+2+9=1+3+8=1+4+7=1+5+6=2+2+8=2+3+7=2+4+6=2+5+5=3+3+6$
$=3+4+5=4+4+4$

5. Given that the sum of N consecutive numbers is 2024, find the maximum number of N.

[Answer] 23

[Solution]

$$a+(a+1)+(a+2)+...(a+N-1)=2024$$
$$\frac{(a+a+N-1)N}{2}=2024$$
$$(2a+N-1)N=4048$$

Since 4048 is an even number and we know $\text{even}=\text{even}\times\text{odd}$, because the average of even number of consecutive numbers is not even, with $4048=2^4\times11\times23$, the answer is 23.

6. Given that x, y, z are positive integers. If $x+\frac{y}{z}=15$ and $y+\frac{x}{z}=20$, find the value of $\frac{x+y}{z}$.

[Answer] 5

[Solution]

$$x+\frac{y}{z}+y+\frac{x}{z}=35$$

$$x+y+\frac{x+y}{z}=35$$

$$\left(\frac{x+y}{z}\right)(z+1)=35$$

We check $z+1$ with factors of 35, to find that when $z+1=7$, $z=6$ and $x+y=30=12+18$. Check: $12+\frac{18}{6}=15$ and $18+\frac{12}{6}=20$. So, $\frac{x+y}{z}=5$.

7. Given that $\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\ldots+\frac{1}{2024^2-1}=A$ where A is in the simplified form, find the sum of the digits of the numerator and denominator of A.

[Answer] 13

[Solution]

$$\frac{1}{(2+1)(2-1)}+\frac{1}{(4+1)(4-1)}+\frac{1}{(6+1)(6-1)}+\ldots+\frac{1}{(2024+1)(2024-1)}$$

$$=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{2023\times 2025}$$

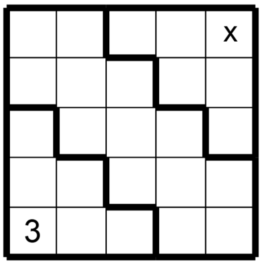
$$=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{2023}-\frac{1}{2025}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{2025}\right)$$

$$=\frac{1012}{2025}$$

The answer is $1+0+1+2+2+0+2+5=13$

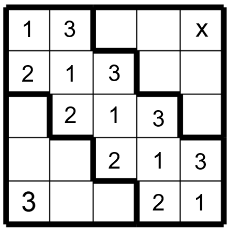
8. Using only the numbers 1, 2, 3, 4, 5, fill in the grid below, ensuring that each number appears only once in each column and row. Given that the sum of the numbers in the three highlighted areas is the same, determine the value of x.



[Answer] 2

[Solution]

The sum of all numbers in this grid is $1\times5+2\times5+3\times5+4\times5+5\times5=75$, so the sum of each highlighted area must be $75\div3=25$. The middle-highlighted area has the sum of 25. We construct this with the extreme case:



So, x must be 2.

9. If [1, 2, 3, …, n] represents the least common multiple of 1, 2, 3, …, n, then among the calculation results of the following 99 formulae $\frac{[1,2]}{2!}$, $\frac{[1,2,3]}{3!}$, $\frac{[1,2,3,4]}{4!}$, …, $\frac{[1,2,3,\ldots,100]}{100!}$, there are how many different values in total?

(This question can be found in Spring Cup 2020)

4、如果$[1,2,3,\cdots,n]$表示 $1,2,3,\cdots,n$ 的最小公倍数，那么算式$\frac{[1,2]}{2!},\frac{[1,2,3]}{3!},\frac{[1,2,3,4]}{4!},\cdots,\frac{[1,2,\cdots,100]}{100!}$的计算结果中共有________个不同的值.

[Answer] 75

[Solution]

If $(n+1)$ is a prime number, it follows that

$$\frac{[1,2,3,\ldots,(n+1)]}{(n+1)!}=\frac{(n+1)\times[1,2,\ldots,n]}{(n+1)\times n!}=\frac{[1,2,\ldots,n]}{n!}$$

From 2 to 100, there are 25 prime numbers. Except 2, because $\frac{[1,2]}{2!}$ is the 1st value.

$99-24=75$

10. $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{31}\right)+\left(\frac{2}{3}+\frac{2}{4}+\ldots+\frac{2}{31}\right)+\ldots+\left(\frac{29}{30}+\frac{29}{31}\right)+\frac{30}{31}=$

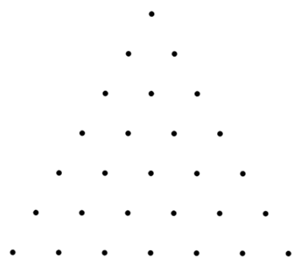
[Answer] 232.5

[Solution]

We can rearrange this to:

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\ldots+\left(\frac{1}{31}+\frac{2}{31}+\ldots+\frac{30}{31}\right)$$
$$=\frac{1}{2}+1+\frac{3}{2}+\ldots+\frac{30}{2}$$
$$=(1+30)\times 15\times\frac{1}{2}$$
$$=232.5$$

11. In the figure below, how many line segments can pass through exactly 3 points?

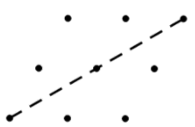


[Answer] 69

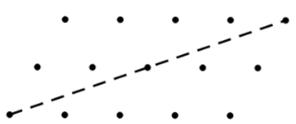
[Solution]

Case 1: 1+2+3+4+5 = 15, with three directions, a total of 45

Case 2: In one direction, there are 6, with three directions, a total of 18



Case 3: In one direction, there are 2, with three directions, a total of 6



Three cases add to 69.

12. How many digits are there in the number 816243240...20162024 ?

[Answer] 875

[Solution]

One digit number: 8 (1 digit)

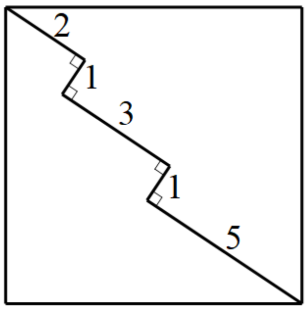
Two digits number: 8 x 2 to 8 to 12 (11 x 2 = 22 digits)

Three digits number: 8 x 13 to 8 x 124 (112 x 3 = 336 digits)

Four digits number: 8 x 125 to 8 x 253 (129 x 4 =516 digits)

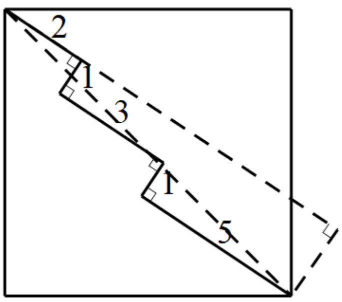
This adds to 875.

13. In the figure below, a square is depicted with line segments drawn inside, each labeled with its length. Determine the area of this square.



[Answer] 52

[Solution]



The square of the diagonal is 10 x 10 + 2 x 2 = 104. So the area is 104 ÷ 2 = 52.

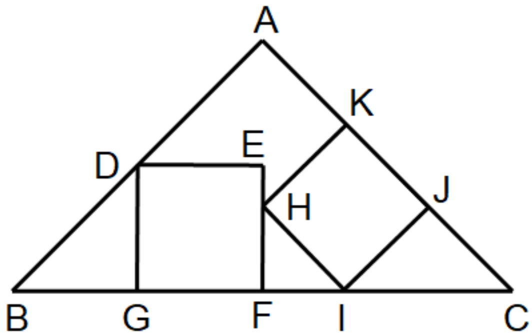
14. There is a sequence 4, 9, 24, 69, 204, 609, x, y, what is the sum of x and y?

[Answer] 7293

[Solution]

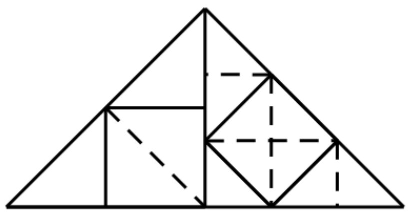
The difference between adjacent numbers are 5, 5x3=15, 15x3=45, 45x3=135, 135x3=405, 1215, 3645, respectively so x is 1824 and y is 5469, the sum is 7293.

15. In the figure below, ΔABC is isosceles right-angled triangle and F is the mid-point of BC. Given that DEFG and KJIH are squares and the area of square DEFG is 126, find the area of the square KJIH.



[Answer] 112

[Solution]



From the figure above, it can be seen 9 units of area corresponds to $126 \times 2 = 252$. So 1 unit is 28 and 4 units is 112.

16. A juice consists of a blend of fruit and water. A restaurant is offering three bottles of juice with equal volumes but varying ratios of water to fruit: 2 : 1, 4 : 3, and 7 : 5 respectively. What would be the resulting ratio of water to fruit if these three bottles of juice were mixed together?

[Answer] 17 : 11

[Solution]

The sum of the three ratios are 3, 7 and 12 respectively. The lowest common multiple of 3, 7 and 12 is 84. So we multiply the three ratios with 28, 12 and 7 respectively, to get 56 : 28, 48 : 36 and 49 : 35. The ratio of water to fruit when combined is (56+48+49) : (28+36+35) = 153 : 99 = 17 : 11.

17. At least how many numbers must be chosen at random from 1 to 19 to ensure the sum of two numbers is 20?

[Answer] 11

[Solution]

In the worst-case scenario, we choose 10 numbers from 1 to 10 where any of the two numbers cannot add to 20. With one more number from the remaining, we can make sure two numbers sum up to 20.

18. In a school band, there are 30 more boys than girls in this year 2023. In year 2024, some new members joined such that the total number of students is 10% more than 2023. The number of boys increased by 5% while the number of girls increased by 20%. How many members were there in 2024?

[Answer] 99

[Solution]

In 2023, the ratio of boys to girls is u+30 to u with a total of 2u+30. In 2024, the ratio of boys to girls is 1.05(u+30) to 1.2u with a total of 1.1(2u+30).

From the question we know 1.05(u+30)+1.2u = 1.1(2u+30), which gives u=30. So the total number in 2023 is 30+30+30=90 and in 2024 is 1.1x90=99.

19. Alice, Bob and Charlie have some marbles. If Alice gives Charlie 20 marbles, the ratio of Charlie's marbles to Alice and Bob's marbles is 2 : 1. If Alice gives Bob 30 marbles, the ratio of Bob's marbles to Alice and Charlie's marbles is 3 : 1. How many marbles do they have in total?

(This question can be found in Spring Cup 2020)

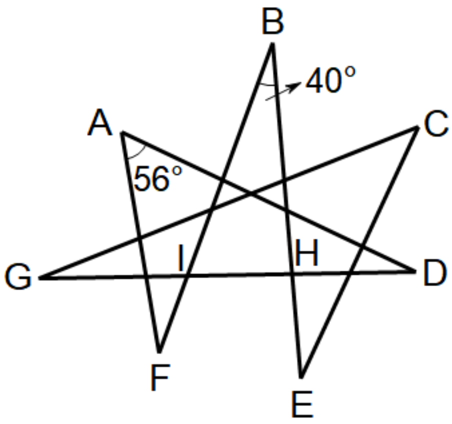
4、甲、乙、丙三人的糖果颗数都是正整数．
如果甲给乙 20 颗糖果，乙的糖果数将是甲、丙糖果数之和的 2 倍；
如果甲给丙 30 颗糖果，丙的糖果数将是甲、乙糖果数之和的 3 倍.
请问：甲、乙、丙三人分别有多少颗糖果？
【答案】甲 30、乙 12、丙 6 【供题】付谦、胡浩、饶海波

[Answer] 48

[Solution]

We know: $\frac{B+A-20}{C+20}=\frac{1}{2}$ and $\frac{A-30+C}{B+30}=\frac{1}{3}$, from the first equation we get $2B+2A-40=C+20$ which can be simplified to $2B+2A-C=60$ and from the second equation we get $3A-90+3C=B+30$ which can be rewrite to $6A+6C-2B=240$. Adding this to the first equation, we get $8A+5C=300$. Since 5C and 300 are both multiple of 5, 8A must also be a multiple of 5. And $A\geq 30$ So, A is 30 or 35. If A=35, we can't find B. So A is 30, B is 6 and C is 12, the sum is 48.

20. As shown in the figure below, find $\angle C+\angle D+\angle E+\angle F+\angle G$.



[Answer] $164°$

[Solution]

From the “arrow head model”:

$\angle G+\angle C+\angle E=\angle GHE$
$\angle A+\angle D+\angle F=\angle FID$
$\angle GHE+\angle FID=180°+\angle B$

So, the answer is $180°+40°-56°=164°$