APMOPS 2024 Round 1

(Provided by Kangaroo Study students)

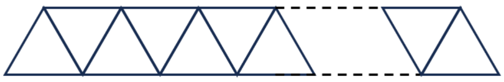
1. In a shop, the price of some item decreased by 5% at first, then it increased by 40% and now it is $990 more expensive than the original price. What was the original price of this item in dollars?
   [Solution] Assuming the original price of the item is 100u,

   100u ×(1 - 5%) × (1 + 40%) = 133u. 1u : 990 ÷ (133-100) = 30,

   then the original price is: 30 × 100 = 3000.

   [Answer] 3000

2. As shown in the diagram below, a pattern of triangle is constructed using matchsticks. If a total of 51 sticks were used, how many triangles are there in the pattern?



   [Solution] Ignoring the leftmost triangle, to form an additional triangle, we will need 2 more sticks. The number of triangles: $(51-3)\div 2+1=25$.

   [Answer] 25

3. During Chinese New Year, Mr. Lee sets up a stack of oranges in a pyramid-like formation where square base is made of 49 oranges (7 by 7). Each orange above the first level rests in the pocket formed by four oranges in the level below. The stack is completed by a single orange in the 7th level. How many oranges are there in this stack?
   [Solution]

   Each layer is a square, with the largest being 7 x 7 and the smallest being 1 x 1, so we add it from the square of 7 to the square of 1.

   $7^2+6^2+5^2+4^2+3^2+2^2+1^2=140$

   [Answer] 140

4. How many distinct triangles are there with perimeter equal to 15 and the three sides are all integer?
   [Solution]

   Count in order based on the sum of the smaller two sides of a triangle being greater than the third side.

   1+7+7, 2+6+7, 3+5+7, 3+6+6, 4+4+7, 4+5+6, 5+5+5

   [Answer] 7

5. Given that 2024 is divisible by 23, find the total number of positive integers x such that 2024 is divisible by x and x is divisible by 23.

[Solution]

In the factors of 2024, to find a multiple of 23, we can find how many factors are left after removing the prime factor 23.

$2024 = 23 \times 88 = 23 \times 2^3 \times 11$

$(3+1) \times (1+1) = 8$

[Answer] 8

6. Five integers are represented by a, b, c, d and e. The average of a, b and c is 100. The average of a, b, c, d and e is 90. Given that e is four times of d, find the value of e.

[Solution]

$a + b + c = 100 \times 3 = 300$

$a + b + c + d + e = 90 \times 5 = 450$

$d + e = 450 - 300 = 150$

$e = 150 \div 5 \times 4 = 120$

[Answer] 120

7. 30 identical small cubes are placed together to form a rectangular solid. Find the total number of different rectangular solids that can be formed in this way.

[Solution]

List three edges in order

$30 = 1 \times 1 \times 30$

$30 = 1 \times 2 \times 15$

$30 = 1 \times 3 \times 10$

$30 = 1 \times 5 \times 6$

$30 = 2 \times 3 \times 5$

[Answer] 5

8. A circus has an acrobatic bicycle with front and back wheels of different sizes. The diameter of the front wheel is 140 cm and the diameter of the rear wheel is 90 cm. How many centimeters did the bike advance when the rear wheel made 10 more revolutions than the front wheel? (Take $\pi = \frac{22}{7}$)

[Solution]

Let the front wheel make x revolutions,

$$140x = 90(x+10)$$

$$x = 18$$

$$140 \times 18 \times \frac{22}{7} = 7920\text{cm}$$

[Answer] 7920

9. Let a, b, c be three distinct positive integers who product abc is equal to 2024. What is the smallest possible value of the sum a + b + c?

[Solution]

When the product is fixed, if the difference is small, then the sum is small.

$$2024 = 8 \times 11 \times 23$$

$$8 + 11 + 23 = 42$$

[Answer] 42

10. Given that p and q are prime numbers such that $3p^2q + 2pq^2 = 483$. Find the largest possible value of $p+q$.

[Solution]

After taking out the common factor of the LHS, consider the case of decomposing 483 into the prime factors.

$$pq(3p+2q) = 3 \times 7 \times 23$$

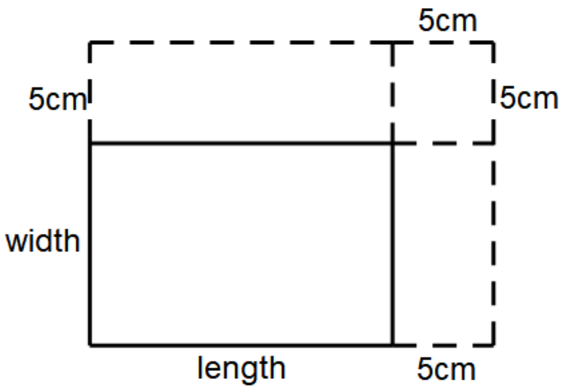
$$p = 3, q = 7$$

$$3 + 7 = 10$$

[Answer] 10

11. If we increase the length and width of a rectangle by 5cm each, the area of the rectangle will increase by 200cm^2. Find the perimeter of the original rectangle in cm.

[Solution]



$l + w : (200 - 25) \div 5 = 35\text{cm}$

P: $35 \times 2 = 70\text{cm}$

[Answer] 70

12. If an n-sided polygon has its sum of interior angles smaller than $2024°$, what is the largest possible value of n?

[Solution]

According to the inner angles of the polygon and the formula,

$2024° \div 180° = 11\ R44°, 11 + 2 = 13$

[Answer] 13

13. Dylan, Bryan and Sheldon split $1300 among them to do investment in different ways. Each begins with a different amount. At the of one year, they now have a total of $1490. It is known that Dylan and Bryan's money has grown by 25% while Sheldon has unfortunately lost $50 in his investment. What was Sheldon's original amount of money at the start of the year?

[Solution] $1490 - 1300 + 50 = 240$, $240 \div 25\% = 960$, $1300 - 960 = 340$.

[Answer] 340

14. Find the smallest positive integer n such that the product of n, n+1 and n+2 is a multiple of 2024.
[Solution] $2024 = 8 \times 11 \times 23 = 4 \times 22 \times 23$

23 cannot become smaller, consider it as one of the numbers.

$21 \times 22 \times 23 (\times)$, $22 \times 23 \times 24 (\surd)$

[Answer] 22

15. Yuhan wants to buy 4 donuts from a shop with sufficient supply of three flavors of donuts: original, chocolate and strawberry. How many possible combinations are there?
[Solution]

Divide the four donuts into three flavors, considering there can also be 0 donut for one flavor, so 4+3=7.

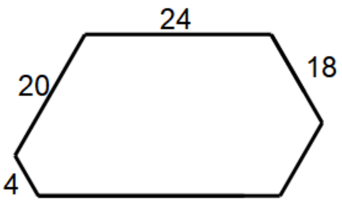
By using the “partitioning method or board-inserting method”, we use 2 boards to split 7 into 3 parts.

$4 + 3 = 7$, $C_6^2 = \frac{6 \times 5}{2} = 15$

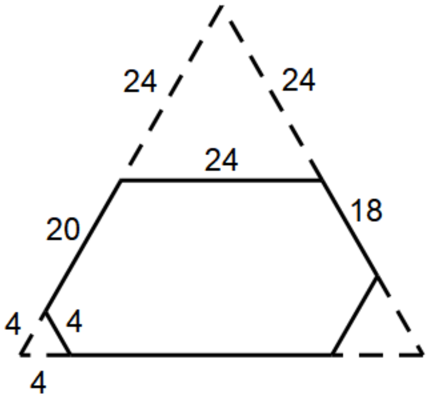
[Answer] 15

(Question 16 onwards may not adhere to correct sequence as presented in the exam.)

16. In the following hexagon, all the interior angles are 120° and some of the sides are given, find the perimeter of this hexagon.



[Solution] Because all interior angles are 120 degrees, lines can be extended outward to form three equilateral triangles.



$24+20+4-24-18=6$
$24+20+4-4-6=38$
$24+20+4+38+6+18=110$

[Answer] 110

17. Given that $\overline{SMOPS}+\overline{APMOPS}=808182$, find $A+P+M+O+S$.
[Solution] S=1(can't be 6), P=9, O=0, M=4, A=7, $1+9+4+7=21$

[Answer] 21

18. How many numbers between 1 and 2024 have a digit sum equal to 10?
[Solution]

__ __ __, $10+3=13$, $C_{12}^{2}=66$, $66-3=63$

1__ __ __, $10-1+3=12$, $C_{11}^{2}=55$

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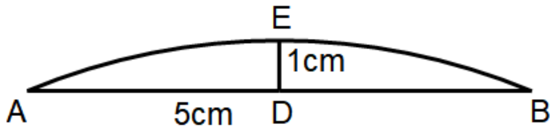
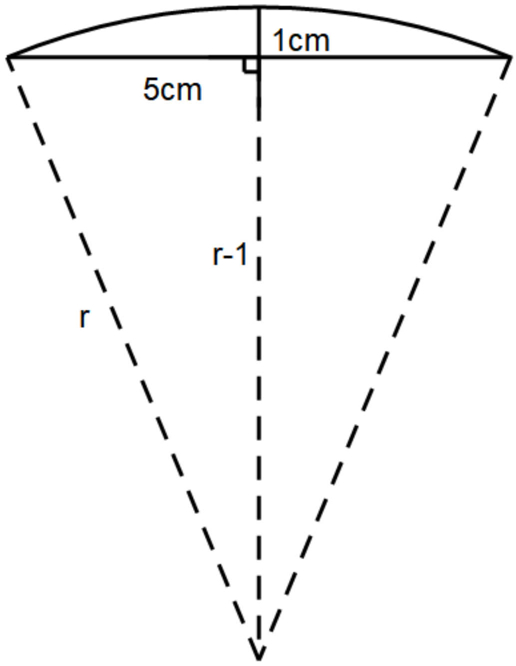
Total: $63+55+2=120$

[Answer] 120

19. AB is an arc segment within a circle, with D as the midpoint of straight line AB, and E as the midpoint of arc AB. Determine the diameter of the circle.



[Solution]



$r^2=(r-1)^2+5^2$, $r=13\text{cm}$, $d=26\text{cm}$

[Answer] 26

20. If $\frac{n}{40-n}$ is a square number, find how many possible values of n are there?

[Solution] Since $\frac{n}{40-n}=\frac{n-40+40}{40-n}=\frac{40}{40-n}-1$ must be an integer, $\frac{40}{40-n}$ must be a factor of 40.

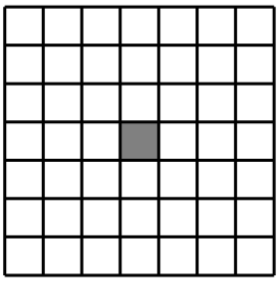
Factor of 40: 1, 2, 4, 5, 8, 10, 20, 40

$\frac{40}{40-n}$	1	2	4	5	8	10	20	40
$\frac{40}{40-n}-1$	0	1	3	4	7	9	19	39

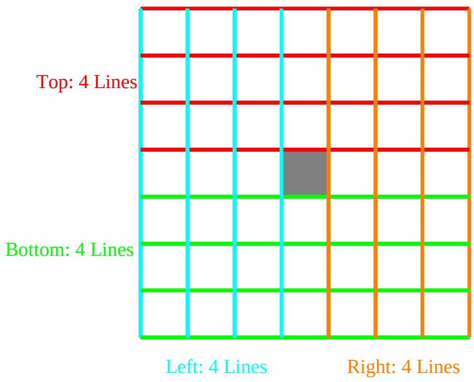
We can get n = 0, 20, 32, 36. Totally 4 possible values.

[Answer] 4

21. In the following grid, how many rectangles include the shaded square and area is even?



[Solution]



Forming a rectangle needs 1 Top Line, 1 Bottom Line, 1 Left Line and 1 Right Line.

Rectangles include the shaded square: $4\times4\times4\times4=256$

Area of Rectangle = length $\times$ width, if area is odd, then length and width must be both odd.

Rectangles whose area is odd: $4\times2\times4\times2=64$

Top Line: one of the 4 red Lines

Bottom Line: 2 Lines (E.g. The first Top Line can be paired with the 2nd or the 4th Bottom Line such that the width is odd)

The case of the Left Line and Right Line are similar to Top Line and Bottom Line.

So, there are $4\times2\times4\times2=64$ rectangles whose area is odd.

Finally, the rectangles include the shaded square and area is even: $256-64=192$.

[Answer] 192

22. P is the product of all the divisors of 2024. How many digits does P have?
[Solution]

$2024=2^3\times11\times23$

$\left(2^0,2^1,2^2,2^3\right)\left(11^0,11^1\right)\left(23^0,23^1\right)$

$4\times2\times2=16$

So, there will be $16\div2$=8 groups factors, whose product is 2024.

$2000^8=256\times10^{24},24+3=27$

[Answer] 27

23. There are 2000 cards in a deck numbered 1 to 2000 respectively. In one operation, all the cards with numbers that are perfect squares are removed. Ten all of the remaining number are renumbered 1, 2, … How many operations must be done so that 1 card remains?

[Solution]

$44^2=1936, 43^2=1849, 42^2=1764$

2000-44=1956, 1956-1936=20

1956-44=1912

1912-43=1869, 1869-1849=20

1869-43=1826

1826-42=1784, 1784-1764=20

$n^2+20-n-(n-1)=n^2-2n+1+20=(n-1)^2+20$

We will minus each number from 2 to 44 two times, but 20 will be subtracted 3 times and 1 will be subtracted one time.

(44-1)×2+1+1=88

[Answer] 88

24. When a number is divided by 100, quotient is q and remainder is r. If the sum of q and r is a multiple of 11, how many four-digit numbers satisfy this?

[Solution]

Let the 4-digit number be $\overline{ABCD}$, so q=$\overline{AB}$, r=$\overline{CD}$,

$\overline{AB}+\overline{CD}$ is a multiple of 11, so $\overline{ABCD}$ is a multiple of 11

1000÷11=90R10

9999÷11=909

909-90=819

(If r can't be 0, $\overline{ABCD}$ can't be 1100, 2200, … 9900, so 819-9=810)

[Answer] 810 (819 if r can be 0)

25. Two farmers sold 400 eggs. They had a different number of eggs at first, and sold them at different prices, but they earned the same amount of money. The first farmer told the second farmer, “If I sold all my eggs with the price of your eggs, I would earn $270.” The second farmer told the first farmer, “If I sold all my eggs with the price of your eggs, I would earn $120.” How much money did the farmers earn altogether?

[Solution]

	A	B
number	x	400-x
price	a	b

ax=b(400-x), bx=270, a(400-x) =120.

So, $b=\frac{270}{x}$, $a=\frac{120}{400-x}$

$$\frac{120x}{400-x}=\frac{270(400-x)}{x}$$

$$\frac{x^2}{(400-x)^2}=\frac{270}{120}=\frac{9}{4}$$

$$\frac{x}{400-x}=\frac{3}{2}$$

$$x=400\times\frac{3}{2+3}=240$$

$$b=\frac{270}{240}=\frac{9}{8}$$

$$(400-240)\times\frac{9}{8}\times 2=360$$

[Answer] $360

26. Two travelers had a total of 77kg of luggage. The first traveler paid $14 for his excess luggage and the second paid $20 for his excess luggage. Had all the luggage belongs to one person, the excess luggage charge would have been $94. What is the maximum luggage weight?

[Solution] Let the maximum luggage weight be x kg.

$(77-2x):(77-x)=(14+20):94$

Solving the equation gives $x=30$

[Answer] 30

27. What is the maximum number such that if you roll a dice, the product of the 5 faces showing is divisible by the number all the time?

[Solution] The HCF of $1\times2\times3\times4\times5$, $1\times2\times3\times4\times6$, $1\times2\times3\times5\times6$, $1\times2\times4\times5\times6$, $1\times3\times4\times5\times6$ and $2\times3\times4\times5\times6$ is 12.

[Answer] 12

28. How many 4-digit numbers with at least two “2” s and one “4”?

[Solution] Case 1: three “2” s and one “4”

2224, 2242, 2422, 4222, totally 4 numbers.

Case 2: two “2” s, one “4” and one random number.

Firstly, “4” can be placed at the ones-place, tens-place, hundreds-place or thousands-place. 4 cases.

Secondly, the one random number can be placed at the remaining three places. 3 cases. And since this digit can be 0, 1, 3, 5, 6, 7, 8, 9, totally 8 cases.

But “0” can’t be placed at the thousands-place, we need remove these cases, 0422, 0242, 0224, totally 3 numbers.

So there will be $4\times3\times8-3=93$ numbers contain two “2” s, one “4” and another digit.

Case 3: two “2” s and two “4”.

2244, 2424, 2442, 4224, 4242, 4422, totally 6 numbers.

In total, $4+93+6=103$ numbers.

[Answer] 103

29. Given that the 49^{th} day of this year is a Monday and the 94^{th} day of the next year is a Saturday. Find the what day is the 1^{st} day of the previous year.

[Solution] If this year is a common year:

Number of days from the 49^{th} day of this year to the 94^{th} day of the next year: $365-49+94=410$ days.

$410\div 7=58R4$, if the 49^{th} day of this year is Monday, the 94^{th} day of the next year will be a Friday. Not matching the condition.

So, this year is a leap year and the previous year is a common year.

Number of days from the 1^{st} day of the previous year to the 49^{th} day of this year: $365-1+49=413$ days.

$413\div 7=59$, so, the 1^{st} day of the previous year is also a Monday.

[Answer] Monday

30. Eight six-face dices were rolled. How many combinations of numbers are there? Order does not matter. (12345666 and 66654321 is the same)

[Solution]

Because there is no difference in the order of number exchange in each case, it is only necessary to consider how many dice are in 1-6.

It can be considered that dividing the 8 dice into 1, 2, 3, 4, 5, and 6. There are no dice that are 1-6, so 8+6=14.

By using the “partitioning method or board-inserting method”, we insert 5 boards into 13 spacings:

$$C_{13}^{5}=\frac{13\times 12\times 11\times 10\times 9}{5\times 4\times 3\times 2}=1287$$

[Answer] 1287